Notes on "Topoi: The Categorial Analysis of Logic"

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Contents

Notes on "Topoi: The Categorial Analysis of Logic" by Robert Goldblatt.
Chapters 1, 2 and 3 are basic set and category theory.

Chapter 3: Arrows Instead of Epsilons

I simply note a couple of the concepts and theorems mentioned that I haven't encountered before this book.

Equalizers

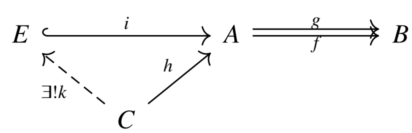
When talking about two set maps $f, g : A \rightrightarrows B$ it makes sense to ask where such parallel arrows agree

$$E = \{a \in A : f(a) = g(a)\}$$

Then there is a natural inclusion $i : E \to A$ that we call the equalizer of f and g. Note that the function is the equalizer not the set, however the function contains the information of the set as its domain. It is clear then that this equalizer does what it claims by making f and g equal under composition

$$f \circ i = g \circ i$$

Then we can also notice that i is the "universal" function with this property. i.e. if we have any other function $h : C \to A$ satisfying $f \circ h = g \circ h$ then h factors uniquely through i. In diagrams



Now we have suficiently abstracted away from the set E and replaced it with a description entirely dependent on the functions. Thus we can give a categorical definition

Definition: *An arrow $i : e \to a$ in a category $\mathscr{C}$ is an equaliser of a pair of parrallel arrows $f, g : a \to b$ iff*

- $f \circ i = g \circ i$
- *Every other morphism with this property factors uniquely through i*

Some useful facts about equalisers are

- All equalisers are monic
- An epic equaliser is an isomorphism

Pullbacks and Completness

A category is finitely complete if it has a limit for every finite diagram (finite number of objects and arrows). An interesting theorem is then that

Lemma. *If $\mathscr{C}$ has a terminal object and a pull back for each pair of arrows (with a common co-domain) then $\mathscr{C}$ is finitely complete.*

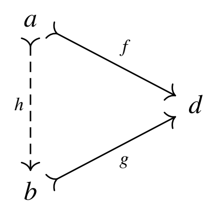
A finitely complete category with exponentiation is Cartesian Closed.

Examples of Pullbacks and Pushouts

Chapter 4: Introducing Topoi

Subobjects

Subobjects will be morphisms that attempt to "categorify" the consept of a subset. First we think of a "subobject" of an object $d \in \mathscr{C}$ as a monic arrow with codomain d, this generalises as the idea of a subset and the (monic) inclusion map. We can think of an inclusion relation between subobjects $f : a \to d, g : b \to d$ as $f \subseteq g$ iff there is an arrow (neccessarily monic) from $a \to b$ such that the diagram commutes.



This is a natural way to generalise the idea of subsets of subsets etc. Note that just as in set, if for two "subobjects" $f \subseteq g$ and $g \subseteq f$ then they have isomorphic codomain and we declare them equal under the equivilence relation $\simeq$ (one has to indeed check that this forms an equivilence relation).

> ***Definition:*** *A subobject of an object $d \in \mathscr{C}$ is an equivilence class of monics with codomain d. Two monics are equivilent if they have isomorphic domains.*
>
> $$sub(d) = \{[f] : f \text{ is a monic with codomain } d\}$$

We carry over the partial order on our informal "subobjects" to the equivilence classes by saying that

$$[f] \subseteq [g] \; iff \; f \subseteq g$$

(one can check that this is well defined) So subobjects form a natural poset in this way just as in set ordered by inclusion.

Elements

In a category with a terminal object, $\mathscr{C}, 1$, then an element of an object, $a \in \mathscr{C}$, is an arrow $x : 1 \to a$ (an element is always a monic). This definition makes sense from the point of veiw we have developed because the element is a subobject with a "singleton" domain.

Subobject Classifier

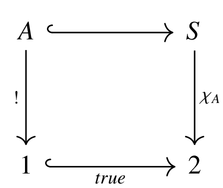
In set we see the notation $\mathcal{P}(S) = 2^S$ because the power set and the collection of functions $S \to \{0, 1\}$ are isomorphic (bijective) in set. The isomorphism is given by considering for each subset $A \subseteq S$ the characteristic function $\chi_A : S \to 2$ sending

$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases} \tag{1}$$

And one shows that the assignment of characteristic functions to subsets is a bijection.

Lemma. *Preimages are pullbacks in Set.*

Thus the following is a pullback square



where *true* : $1 = \{0\} \to 2 = \{0,1\}$ sending $0 \mapsto 1$. χ_A is the one and only one function making this a pullback diagram, thus we have given a more general categorical description of the characteristic function that we will use to define subobject classifiers.

> ***Definition:*** *A subobject classifier for a category $\mathscr{C}$ with a terminal object 1 is an object Ω and an arrow true : $1 \to \Omega$ satisfying the Ω-axiom: For each monic $f : a \to d$ there is one and only one arrow $\chi_f : d \to \Omega$ such that*
>
> $$\begin{array}{ccc} a & \hookrightarrow & d \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\hookrightarrow} & \Omega \end{array}$$
>
> *is a pullback square.*

χ_f is the characteristic arrow (or character) of f. We often denote *true* by $\top$. The subobject classifier is unique up to isomorphism.

Recalling the equivilence relation on monics $\simeq$ we now justify the foregoing discussion with the theorem

Theorem. *Given two monics $f : a \to d$ and $g : b \to d$ then*

$$f \simeq g \iff \chi_f = \chi_g$$

Elementary Topos

> ***Definition:*** *An elementary topos is a category $\mathscr{E}$ such that*
>
> 1. *$\mathscr{E}$ is finitely complete*
> 2. *$\mathscr{E}$ is finitely cocomplete*
> 3. *$\mathscr{E}$ has all exponentials*
> 4. *$\mathscr{E}$ has a subobject classifier*

Note that this definition is overkill, namely condition 3 is implied by the others. Further the existence of exponentials also implies that the category has products.

1 can be replaced by

$(1')$ $\mathscr{E}$ has a terminal object and pullbacks

while 2 can be replaced by

$(2')$ $\mathscr{E}$ has an initial object and pushouts

Power Objects

These are the categorical generalisation of the power set. The key idea is that in Set there is an isomorphism

$$\{\text{functions } B \to \mathcal{P}(A)\} \overset{\sim}{\leftrightarrow} \{\text{relations } B \to A\}$$

The bijection is given a function $f : B \to \mathcal{P}(A)$ we define the relation $R_f \subseteq B \times A$ by

$$xR_f y \iff y \in f(x)$$

given the relation we retrieve the function by letting

$$f_R(x) = \{y \in A : xRy\}$$

We also have the Set isomorphism from $\mathcal{P}(A) \to 2^A$ and we can observe that composing these ideas gives that

$$\in_A \cong \{(\chi_U, x) : U \subseteq A, x \in A, \chi_U(x) = 1\} = \epsilon'_A \subseteq 2^A \times A$$

where $\in_A$ is the membership relation between $\mathcal{P}(A)$ and A. The characteristic function of the set ϵ'_A is the evaluation arrow

$$ev : 2^A \times A \to 2$$

This gives us the categorical categorisation of ϵ'_A as the pullback square and so in general (one has to prove this) given a relation $R \subseteq B \times A$ then f_R is the unique function giving a pullback of the following form (regardless of what g is)

pullback square

pullback diagram

> ***Definition:*** *A category, $\mathscr{C}$, with products is said to have power objects if for any $a \in \mathscr{C}$ there are $\mathscr{C}$ objects $\mathcal{P}(a)$, $\in_A$ and a monic arrow $\in : \in_a \to \mathcal{P}(a) \times a$ such that for any $b \in \mathscr{C}$ and any monic $r : R \to b \times a$ there is exactly one arrow $f_r : b \to \mathcal{P}(a)$ for which there is a pullback of the form*

pullback diagram

Theorem. *Any Topos has power objects*

Indeed another definition of a Topos may be a category that is finitely complete and has power objects.

Subobject Classifiers and Comprehension

Examples

Categories of sheaves and bundles give important examples.

Chapter 5: Topos Structure First Steps

Now that we have the concept of a topos in hand we should explore the basic properties that it has.

Monics Equalise

- Monics are equalisers for their characteristic functions.
- Any arrow is an iso iff it is both monic and epic in a topos.
- $\mathit{true} : 1 \to \Omega$ equalises $1_\Omega : \Omega \to \Omega$ and $\mathit{true}_\Omega : \Omega \to \Omega$

Images of Arrows

Any set function can be factored into a surjection (the function onto its image) composed with an injection (the inclusion of the image into the codomain). This is unique up to unique commuting iso. This is also true in any Topos.

Fundamental Facts

- The comma category of a topos is a topos
- Pullbacks preseve epics
- Copordicts preserve pullbacks

This is related to the fundamental theorem of Topoi. More later.

Extensionality and Bivalence

Extensionality Principle for Arrows: If two arrows are distinct they have elements on which they disagree (i.e. 1 is a generator) (note that this is not trivial becuase here element means an arrow out of 1).

- If there is an arrow $x : 1 \to 0$ (an element of the initial) then the topos is degenerate (all objects are isomorphic)
- A nondegenerate topos that satisfies the extensionality principle for arrows is called well pointed.
- Every nonzero objcet is nonempty in a well pointed topos.

In SET there are two arrows from $1 = \{0\} \to \Omega = \{0, 1\}$. True sends 0 to 1 and False sends 0 to 0. False is given by pullback, in particular it is χ_{0_1}, which we denote by $\perp$ in a general topos.

Elements of the subobject classifier are called truth values. A topos is bivalent its only truth values are true and false. If the topos is well pointed then it is bivalent.

In a topos we have coproducts, hence we can take the coproduct $1 + 1$ which in set is isomorphic to $\Omega = 2$ via $[\top, \perp]$ (the coproduct of the true and false maps). If $[\top, \perp]$ is an isomorphism in a topos the topos is called classical. If a topos is well pointed then it is classical.

Theorem. *A topos is well pointed iff it is classical and every nonzero object is nonempty.*

Theorem. *If M is a mointind then the category M-SET is classical iff M is a group.*

This gives an easy way of finding examples of nonclassical topos.

Monics and Epics by Elements

We now think of elements as morphisms from 1 and application of functions as compositions with elements. This means that injectivity and surjectivity are now frased as compositions of morphisms. In a well pointed topos this concept of surjectivity corrosponds with epic and injectivity with monic.

Chapter 6: Logic Classically Conceived

Motivating Topos Logic

In set theory we have the algebra of classes (compliments, intersections and unions) together with powre sets. This structure is that of a boolean algebra. The logical operations are used in defining these sets and this is the reason that they form a boolean algebra. The rules of logic are expressed in SET by operations on $0, 1$ i.e. the elements of 2.

We can abstract these operations in terms of arrows and thereby get a general algebra of subobjects in an arbitrary topos. The algebra may or may not be boolean and this will be a way to differentiate the internal logic of a topos. Likewise the rules of the logic will be expressable on the elements of the subobject classifier of the topos.

Propositions and Truth Values

Propositions are either true or false. We denote true by 1 and false by 0. The connectives on propositions can then be defined by truth tables (their meaning explained). Connectives can be thought of as truth functions $\wedge, \vee, \implies : 2 \times 2 \to 2$ or $\neg : 2 \to 2$.

Propositional Calculus

The propositional calculus is given as a formal language with some alphabet ($\{\pi_0, ...\}$) and formation rules. A semantics is given by a value assignment (a map from the alphabet to 2) that assigns a truth value to the letters of each formula. This is extended to a full semantics via the truth tables (induction over the formation rules). A formula or sentence is a tautology or classically valid if it is true for every assignment possible, denoted $\models \alpha$.

This same class of sentences can be captured with 12 axioms and modus ponens (as the set of provable sentences of a sequent caluculs or such). The statement that this deduction system captures the same formulas as the tautologies are the soundess and completness theorems.

Unironically the first time I have understood semantics.

Boolean Algebra

Recall that a lattics is a poset $(P, \sqsubseteq)$ in which any two elements $x, y \in P$ have a greatest lower bound $x \sqcap y$, and a least upper bound $x \sqcup y$. Respectively known as the meet and join of x and y. When the poset is considered as a category then meets are products and joins are coproducts (hence the notation). A minimim in a lattice is an element $0 \in P$ such that for all $x \in P$ we have $0 \sqsubseteq x$, while a unit or maximum is denoted 1. A lattive is bounded if it has a unit and a zero, categorically an initial and a terminal object. Note that a lattice always has all pushouts and pullbacks making it a finitely bicomplete skeletal pre order category.

A latttice is distributive if it satisfies

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$

or equivilently

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$$

In a bounded lattice y is a compliment of x iff $x \sqcup y = 1$ and $x \sqcap y = 0$. A bounded lattice is complimented if each of its elements has a compliment.

> ***Definition:*** *A boolean algebra is a complemented distributive lattice.*

Algebraic Semantics

The boolean operations allow one to interpret logical formulas as objects in a boolean algebra and the valuation must preserve the logical operation as algebraic structure. Instead of a valuation taking values in 2 it can now take them in an arbitrary boolean algebra B and we say that a sentence is B valid, $B \models \alpha$ iff for every B valuation V the sentence evaluates to 1, the unit of the boolean algebra.

A sentence is BA valid if it is valid in every valuation with codomain any boolean algebra. There is a soundess theorem (provable implies B valid) and surprisingly we have an equivilence

- α is CL provable
- α is a tautology (2 provable)
- There is some B such that $B \models \alpha$
- α is BA valid

Truth functions as arrows

Each truth function (the function $\wedge : 2 \times 2 \to 2$ that represents the truth table for example) how codomain 2 and hence is the characteristic function of some set. This is something that can be generatlised to a topos.

Disjunction, negation, conjunction and implication are then given as arrows in diagrams and then generalised to:

- $\neg : \Omega \to \Omega$ is the unique arrow making the following a pullback

$$\begin{array}{ccc} 1 & \xrightarrow{\ \bot\ } & \Omega \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \neg} \\ 1 & \xrightarrow{\ \top\ } & \Omega \end{array}$$

- $\cap : \Omega \times \Omega \to \Omega$ is the character of the product arrow $(\top, \top) : 1 \to \Omega \times \Omega$
- $\cup : \Omega \times \Omega \to \Omega$ is the characer of the image of the arrow

$$[\langle \top, 1 \rangle, \langle 1, \top \rangle] : \Omega + \Omega \to \Omega \times \Omega$$

- $\implies : \Omega \times \Omega \to \Omega$ is the character of

$$e : A \to \Omega \times \Omega$$

  which is the equaliser of

$$\Omega \times \Omega \underset{\cap}{\overset{\pi_1}{\rightrightarrows}} \Omega$$

  where $\pi_1 : \Omega \times \Omega \to \Omega$ is the first projection

Topos Semantics

We can now do propositional logic in any topos. A truth value is an arrow $1 \to \Omega$. A valuation is then a function $letter \to Hom(2, \Omega)$ which is extended to all sentences by inductively:

- $V(\neg\alpha) = \neg \circ V(\alpha)$
- $V(a \wedge b) = \cap \circ \langle V(a), V(b) \rangle$

etc where the logical operations on the right are the arrows just defined. A sentence is $\mathscr{E} - valid$, denoted $\mathscr{E} \models \alpha$ iff for evey valuaution $V(\alpha) = \top : 1 \to \Omega$.

Any topos is complete wrt classical logic. i.e. any $\mathscr{E}$ valid sentence is a CL theorem. They are not all sound however. In fact the first 11 rules of CL listed by Goldblatt are valid in any Topos, however the 12th, the law of excluded middle may fail in some Topoi.

If the topos is bivalent then it is both sound and complete wrt Classical logic.

Chapter 7: Algebra of Subobjects

Complement, interseciton, union

The link between classifiers and BA can be made explicit in set where

- $\chi_{-A} = \neg \circ \chi_A$
- $\chi_{A \cap B} = \chi_A \cap \chi_B$
- $\chi_{A \cup B} = \chi_A \cup \chi_B$

Now define in a topos the algebra of subobjects (generalising this BA of subsets in SET) to be:

-

write out. They are the natural generalisation of the above though.

these operations turn out to be unique in a topos.

Sub(d) as a lattice

Now we have defined some operations on the collection of subobjects of something in a topos. These operations do indeed form a lattice just as in SET. In fact the operations defined form a bounded distributive lattice with $f \cap -f$ always zero. The last peice to make it a boolean algebra would be $f \cup -f$ being the unit, however this fails in some topoi. The following theorem helps generate examples of non-boolean topoi

Theorem. *In any topos is $\top : 1 \to \Omega$ has a compliment in $Sub(\Omega)$ then this compliment is $\bot : 1 \to \Omega$*

Boolean topoi

A topos is boolean if for every object $(Sub(d), \subseteq)$ is a boolean algebra. The following are then equivilent

- The topos is boolean
- $Sub(\Omega)$ is a BA
- $\top : 1 \to \Omega$ has a complement in $Sub(\Omega)$ (if it exists it is $\bot$)
- $\top \cup \bot \simeq 1_\Omega$ in $Sub(\Omega)$
- The topos is classical
- $\iota_1 : 1 \to 1 + 1$ is a subobject classifier.

internal vs External

Theorem. *If a topos $\mathscr{E}$ is boolean then for any sentence α we have $\mathscr{E} \models \alpha \vee \neg\alpha$*

Note that the converse does not hold, there are examples of topoi with excluded middle who are not Boolean. The relevant condition is actually:

- $\mathscr{E} \models \alpha \iff \alpha$ is provable for all sentences α.
- $\iff \mathscr{E} \models \alpha \vee \neg\alpha$ for all α
- $\iff Sub(1)$ is a BA

Something subtle is happening here with how the logic interacts with $Sub(1)$. The $Sub(d)$ is a collection of subobjects of d, however it may not be itself an object of the topos (as it is in SET). This is the reason for the exponentials Ω^d or the internal Homs.

The notion of excluded middle can be written internally in SET by saying $x \cup \neg x = 1$ for $x \in 2$ which is equivilent to the commutativity of the following diagram

$$\begin{array}{ccc} \Omega & \xrightarrow[\langle 1_\Omega, \neg \rangle]{} & \Omega \times \Omega \\ \Big\downarrow{\scriptstyle !} & & \Big\downarrow{\scriptstyle \cup} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

However this diagram exists in any Topos. A theorem is then that $Sub(\Omega)$ is a BA iff the above diagram commutes.

This notion of internal vs external can be carried to the semantics and explains why things may behave strangely. When we replace external homs by internal exponentials then they act as we would expect. Thus it is when the INTERNAL logic is classical that the topos is boolean.

Implication and its implications

Chapter 8: Intuitionism and Its Logic

The key difference is that while classical theories assume at the outset that every formula is either true or false, intuitionism reserves that for formulas that have been shown to be true or false.

> “ *Brouwer’s view of the history of logic is that the logical laws were obtained by abstraction of the structure of mathematical deductions at a time when the latter were concerned with the world of the finite. These principles of logic were then ascribed an a priori independent existence. Because of this they have been indiscriminately applied to all subsequent developments, including manipulation of infinite sets. Thus contemporary mathematics is based on and uses procedures that are only valid in a more restricted domain. To obtain genuine mathematical knowledge and determine what the correct modes of reasoning are we must go back to the original source of mathematical truth.* ”

Heyting postulated that this intuitive notion was captured by classical logic with the law of excluded middle removed.

Heyting Algebras

We want to now talk about the least upper bound of a set, not just pairs. We say that a subset of a lattice $A \subseteq (L, \sqsubseteq)$ has an upper bound $x \in L$, denoted $A \sqsubseteq x$ iff for every $y \in A$ we have $y \sqsubseteq x$. If $A \sqsubseteq z$ implies that $x \sqsubseteq z$ then x is a least upper bound. Finally if $x \in A$ then we say it is a greatest element of A.

We say that in a lattice with a 0 element, a is the psuedo-compliment of b iff b is the greatest element of $\{x \in L : a \sqcap x = 0\}$. If every member has a pseudo compliment then L is psuedo-complimented lattice. More generaly we might say that a is the pseudo-compliment of b relative to c iff it is the greatest element of $\{x : b \sqcap x \sqsubseteq c\}$.

In a lattice L the psuedo-compliment of a relative to b, when it exists is denoted $a \implies b$. If this object always exists then we say L is relatively pseudo-complimented.

Some facts

- The Boolean compliment is always a pseudo-compliment.
- $a \sqsubseteq b$ iff $a \implies b = 1$

Definition: *A Heyting algebra is a relatively pseudo-complimented (r.p.c.) Lattice that has a zero object.*

In a Heyting algebra we define $\neg a \equiv a \implies 0$, as the psudo-compliment.

Lemma. *Every rpc lattice is distributive.*

Note that $x \sqsubseteq \neg\neg x$ however the converse is not always true. Indeed it is true for all x only when a Heyting algebra is a Boolean algebra (when the psudeo-compliment is the actual compliment).

Theorem. *In any topos the following are equivilent*

- *The topos is Boolean*
- *In $Sub(\Omega)$, $\top \cong -\bot$*
- $\neg \circ \neg = 1_\Omega$

The definition of a semantics in a Heyting algebra is exactly as it was for a Boolean algebra, we have the theorem that

$$\alpha \text{ is HA-Valid iff } \vdash_{IL} \alpha$$

i.e. semantics in Heyting algebras are sound and complete wrt intuitionistic logic.

We have a bijection

$$Sub(d) \cong \mathscr{E}(d, \Omega)$$

which induces a HA structure on $\mathscr{E}(d, \Omega)$, the structures are given by applying the truth functions as before in a Boolean algebra. From definition this means that the two are also isomorphic as HA. In summary, for a topos $\mathscr{E}$

$$\mathscr{E} \models \alpha \text{ iff } \mathscr{E}(1, \Omega) \models \alpha \text{ iff } Sub(1) \models \alpha$$

Where the second and terd terms are Heyting Algebras. Hence topos validity is essentially just Heyting Validity. This is again given by

$$\underset{IL}{\vdash} \alpha \text{ then } \mathscr{E} \models \alpha \text{ for every topos } \mathscr{E}$$

6.1.1 Exponentials

The implication $a \implies b$ gives the exponential object b^a in a r.p.c. lattice.

> “ *we find that categorially a Heyting algebra is no more nor less than a Cartesian closed and finitely co -complete poset* ”

Kripke Semantics

History Kripke provided a model of IL where sentences are subsets of a fixed poset. This grew out of his analysis of modal logic. Tarski translated IL sentences into modal sentences in such a way that

$$Translation(Sentence) \iff sentence$$

Is this correct?

Interpreting IL then through this translation gives a new way of understanding the sentences (in some sense). In connection with the philosophy of intuitionism as theorems being temporally order, and dependent on human action, this poset setup in modal logic was well suited to match up with an understanding of theorems as possibly true in the future, time giving the poset structure and the idea that once theorems are proved they are true forever, but before that they are undetermined giving the modal nature.

Formalities A poset in this context may also be refered to as a frame. We say a set $A \subseteq (P, \sqsubseteq)$ is hereditary in P if its closed upwards, i.e. $p \in A$ and $p \sqsubseteq q$ then $q \in A$. The hereditary subsets of P are denoted P^+.

A P valuation then is the normal function into P^+. Becasue the sets are hereditary, a formula is always true, for any time after it was proven.

Defining a model, true and valid

If we denote $M(\alpha) = \{p : M \models_p \alpha\}$ then we observe that

- $M(\alpha \wedge \beta) = M(\alpha) \cap M(\beta)$
- $M(\alpha \vee \beta) = M(\alpha) \cup M(\beta)$
- $M(\neg\alpha) = \neg M(\alpha)$

etc indeed the poset $(P^+, \subseteq)$ is a Heyting algebra under the normal lattice operations. Hence a valuation for P is actually a HA-valuation of P^+. Indeed

$$P \models \alpha \text{ iff } P^+ \models \alpha$$

so a sentence is intuitionistically provable iff it is valid for every frame (Soundness and completeness of Kripke semantics).

On any frame P the collection P^+ gives a topology.

Topological interpretation of IL? [68], [69], [70] THomason

6.2.1 Beth Models

Chapter 9: Functors

I will note only concepts that I have not yet covered.

A category is skeletal if objects are isomorphic iff they are equal. Each Category has a full skeletal subcategory, by using choice on the collection of isomorphism equivilence classes (create a new category with only one representative from each class), moreover the skeletal subcategory is equivilent to the whole category via the inclusion map.

There is something weird here because the cardinality of the skeletal category can be vastly different (countable vs proper classes for instance).

thread

Theorem. *For any category $\mathscr{C}$ the category $SET^{\mathscr{C}}$ is a topos*

Its pullbacks are defined component wise, the subobject classifier is

$$\Omega : \mathscr{C} \to SET$$

$$a \mapsto \{S : S \text{ is an a sieve }\}$$

$$f \mapsto \Omega(f)$$

where an a-sieve is a subset of $S_a = \{\mathscr{C}$ arrows that have domain a $\}$ that is closed under left composition. (Very similar to an ideal), note that S_a is always an a-Sieve. $\Omega(f)$ sends the a-sieve S to the b-sieve $\{g : b \to c | g \circ f \in S\}$. With the arrow (natural transformation) $\top : 1 \to \Omega$ with components $\top_a(0) = S_a$ outputting on a the largest sieve.

Chapter 10: Set Concepts and Validity

Set Concepts

In light of the intuitionistc concept of truth as temporal the set comprehension schema is a little weird, because the sets of which a formula φ are true will change in time. We define then

$$\varphi_p = \{x : \phi(x) \text{ is known at time p}\}$$

There is some weird mixing of ontology or something, this is like modelling IL set theory in CL set thoery or something

For a frame P the map $p \mapsto \phi_p$ defines a functor between P and SET, where because truth persists in time the arrow $p \to q$ in P is assigned the inclusion arrow $\varphi_p \hookrightarrow \varphi_q$. So to every ϕ we get a functor, or an object in SET^P. This functor will be called a variable set, intensional set or a set concept.

The philosophical idea at work here is the the functor captures all the "meaning" of φ while the φ_p is the actual object to which we refer in speech.

The remaining part of the chapter is a case study of the topos structure of SET^P. In short its logic is intuitionistic. It is essentially an application of the definitionsn and thoeroms for the general functor categories above however.

Heyting Algebras in P

Truth Arrows

Validity

Applications

$$SET^P \models \alpha \text{ iff } P \models \alpha$$

where the left is topos validity and the right is Kripke validity.

This gives the completenss theorem of Topos semantics

$$\vdash_{IL} \alpha \text{ iff } \alpha \text{ is valid in every topos}$$

8.5.1 LC

LC is the logic given by adjoining the classical tautology

$$(a \implies b) \vee (b \implies a)$$

to IL. This is intermediate between IL and CL in the sense of the collection of theorems being strictly contained between them

$$Th(IL) \subset Th(LC) \subset Th(CL)$$

It is known that

$$\vdash_{LC} \alpha \text{ iff } \mathbb{N} \models \alpha \text{ iff } \mathbb{Q} \models \alpha \text{ iff } \mathbb{R} \models \alpha$$

Then composing with the equivilence above we know that the Topoi $SET^{\mathbb{N}}, SET^{\mathbb{Q}}, SET^{\mathbb{R}}$ all have equivilent logics. In fact

Theorem. *If P is any linearly ordered poset then*

$$SET^P \models \alpha \textit{ iff } \vdash_{LC} \alpha$$

Chapter 11: Elementary Truth

Goldblatt notes that the text will be more conversational from here on.

First Order Theories

Mentions the basics of a first order theory and sets up the arrows only first order theory of categories.

W. S. Hatcher [68] for an extensive elaboration on this first order theory

Formal Languages and Semantics

- model theory classically
- model theory in a topos, so extending the previous notions to the quantifiers
- Extending previous soundness and completeness theorems
- Existence and free logic: What if the domain of the model is empty
- “ *Emerging from this discussion is a generalised concept of a "set" as consisting of a collection of (partial) elements, with some Heyting- algebra-valued measure of the degree of equality of these elements.* ”
- Higher Order Theories:

  Something that jumps out to me is that the fact that it is higher order seems to be a property of the models, not the syntax?

Chapter 12: Categorial Set Theory

Examining the ZFC axioms as modelled in a topos.

Chapter 13: Arithmetic

Examining the Peano axioms as modelled in a topos.

Chapter 14: Local Truth

The aim is to develop the notion of a topology on a category as well as its relevant sheaf concept. Then look at the axiomatic sheaf theory on a topos.

Stacks and Sheaves

We can think of a topological space, $(T, \mathcal{T})$, as a category where objects are elements of $\mathcal{T}$ and arrows are inclusions.

$$Hom(U, V) = \{i : U \to V\}$$

A stack or presheaf is a functor

$$F : \mathcal{T}^{op} \to SET$$

So each open gets a set and each inclusion $U \hookrightarrow V$ gets assigned a function $FV \to FU$. The collection of stacks over a topology $\mathcal{T}$ we denote $St(\mathcal{T})$, and this is by definition the topos $SET^{\mathcal{T}^{op}}$.

The elements of $F(U)$ are called sections. The functions $F(i : U \hookrightarrow V) : F(V) \to F(U)$ are called restrictions, often denoted $Res_{U,V}$ or indeed $|_U$. Hence the restriction of a section $s \in F(V)$ is denoted

$$F(i)(s) = s|_U$$

To make a stack a sheaf we want that gluing sections be consistent, functorially we then require for every open set U and open cover $(U_i)_i$ of U

- For every $s, t \in F(U)$, if for all i $s|_{U_i} = t|_{U_i}$ then $s = t$
- If $\{s_i \in F(U_i)\}_i$ such that for every i,j $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there is some $s \in F(U)$ such that for each i $s|_{U_i} = s_i$

Notice that the collection of sheaves on a topological space forms a full subcategory $Sh(\mathcal{T}) \hookrightarrow St(\mathcal{T})$. There is an equivilence of categories between $Sh(\mathcal{T})$ and $Top(\mathcal{T})$, the category (topos) of sheaves of germs over T.

$$\text{Sheaves of sections} \;\cong\; \text{Sheaves of germs}$$

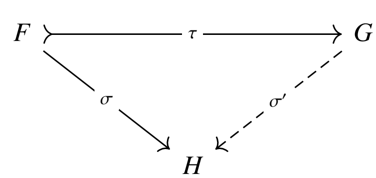
Not clear on what the second category even is, I think the point is that its a topos though.

This poset has a natural numbers object.

As already mentioned for a poset (frame) P the collection of hereditary sets P^+ forms a topology, then a sheaf over this category (P as a topological space) is the same information as a Kripke semantics.

Classifying Stacks and Sheaves

Theorem. *A stack H is a sheaf iff for very arrow $\sigma \in Hom_{St(I)}(F, H)$ and every dense arrow $\tau : F \to G$ there is exactly one $\sigma' : G \to H$ such that the diagram commutes*



Grothendieck Topoi

Sites A pretopology on a category is an assignment to each object, $\alpha \in \mathscr{C}$, a collection of sets of arrows, $Cov(\alpha)$ with codomain α. These collections are called covers of α and must satisfy

- Identity is a cover: $\{id_a\} \in Cov(a)$

- Covers compose: Given $\{b_i \xrightarrow{f_i} a : i \in I\} \in Cov(a)$ and for each i we also have a cover $\{c_{i,j} \xrightarrow{f_{i,j}} b_i : j \in J\} \in Cov(b_i)$ then

$$\{c_{i,j} \xrightarrow{f_i \circ f_{i,j}} a : i \in I, j \in J\} \in Cov(a)$$

- Covers of intersections: Given $\{a_i \xrightarrow{f_i} a : i \in I\} \in Cov(a)$ and an arrow $g : b \to a$ then the following pullback exists and forms a cover of b

$$\begin{array}{ccc} b \underset{a}{\times} a_i & \dashrightarrow & a_i \\ {\scriptstyle g_i} \downarrow & & \downarrow {\scriptstyle f_i} \\ b & \xrightarrow{g} & a \end{array}$$

  and

$$\{b \underset{a}{\times} a_i \xrightarrow{g_i} : i \in I\} \in Cov(b)$$

The category with a given pretopology is called a site.

And indeed when considering a topological space as a category the collection of covers of all objects is a site.

Generalised stacks and sheaves Given a stack (contravariant functor from $\mathscr{C}$ to SET) F over a site $(\mathscr{C}, Cov)$ and a cover $\{a_i \xrightarrow{f_i} a : i \in I\} \in Cov(a)$ then define $F_{x,y} : F(a_x) \to F(a_x \underset{a}{\times} a_y)$ as the F image of the pullback (morphism) of f_x along f_y.

A stack F is a sheaf over the site $(\mathscr{C}, Cov)$ iff for any cover $\{a_i \xrightarrow{f_i} a : i \in I\} \in Cov(a)$ and any $s_i \in F(a_i)$ that are pairwise compatible

$$F_{x,y}(s_x) = F_{y,x}(s_y)$$

then there is exactly one $s \in F(a)$ such that $F(f_x)(s) = s_x$ for every $x \in I$.

Our classification generalises.

> ***Definition:*** *A grothendieck topos is a category that is equivalent to $Sh(\mathscr{S})$ for some site $\mathscr{S}$.*

Elementary Sites

A topology on an elementary topos is an arrow $j : \Omega \to \Omega$ that is a closure operator

- $j \circ true = true$
- $j \circ j = j$
- $\cap \circ (j \times j) = j \circ \cap$

An elementary topos with a topology is called an elementary site.

Geometric Modality

Kripke-Joyal Semantics

Number Systems as Sheaves

Chapter 15: Adjointness and Quantifiers

Defines adjoints. The important thing to notice is that all the topos structures we have looked at are special examples of adjoints.

$$L \dashv R \iff Hom(LA, B) \cong Hom(A, RB)$$

The Fundamental Theorem

Theorem. *For any topos $\mathscr{E}$ and any object $b \in \mathscr{E}$ the comma category $\mathscr{E} \downarrow b$ is a topos and for any arrow $f : a \to b$ we get a functor, $f^* : \mathscr{E} \downarrow b \to \mathscr{E} \downarrow a$ that has both a left and right adjoint*

The section is explaining this result.

Quantifiers

Models of quantifiers are given by adjoints.

Chapter 16: Logical Geometry

Examinging the influence that the logical analysis of topoi can have on geometry (instead of the converse which is the rest of the book).

The goal of the chapter is to present Delignes theorem as a verson of the completeness theorem.

Preservation and Reflection

If P is a categorical property then a functor F preserves P iff the image under F of an object/morphism (depending on the property) has the property P. F reflects P iff when the image of an entity has P then the entity has P.

$$P(a) \implies P(F(a)) \qquad \text{(preserves)}$$

$$P(F(a)) \implies P(a) \qquad \text{(reflects)}$$

A functor is faithful if it is injective on each hom set.

Theorem. *For a functor out of a topos that preserves equalisers and pullbacks we have*

$$faithful \iff conservative \iff preserves\ \ iso's$$

If a functor between topoi is exact (preserves all finite limits and colimits) then it preserves images. Recall that left adjoints are right exact (preserve limits) and right adjoints are left exact (preserve colimits)

Geometric Morphisms

A geometric morphism between elementary topoi is an adjunction of exact functors.

There is a good discussion here and in the adjoint section about pull back morphisms and functors begin adjoints etc Puts the dif top/geo into a very general framework.

Kan Extensions A functor $F : \mathscr{C} \to \mathscr{D}$ lifts to a functor $\bar{F} : St(\mathscr{C}) \to St(\mathscr{D})$ via precomposition. The Kan extension of F is a left adjoint to $\bar{F}$. The functor can be constructed somewhat directly as the limit of a diagram (component wise).

Has some more interesting stuff but nothing too relevant to my current interest

Internal Logic

Gives a very explicit definition of a model of a language in a category.

Definition: *Given $\mathscr{C}$ a finitely complete category we can associate a many sorted language $\mathscr{L}_{\mathscr{C}}$*

- *The sorts are the objects of $\mathscr{C}$ (so even if $\mathscr{C}$ has a proper class of objects we are still making this a language)*

  this is clearly nonsense

- *Each arrow $f : a \to b$ is a one place operation symbol with type (a, b) (product type)*

This language is the internal language of the category.

There is then a canonical model of this language in $\mathscr{C}$ via the identity. The idea is then to characterise properties of $\mathscr{C}$ via the truth of certain equations in $\mathscr{L}_{\mathscr{C}}$. It is noted that the finite completeness is a minimal requirment for the language to contain the equality. A simple example of this is

$$term(a) \equiv (\exists v(v = v)) \wedge (v = w)$$

then

$$\mathfrak{U} \models term(a) \text{ iff a is terminal}$$

Geometric Logic

Theories as Sites